JOURNAL OF COMPUTATIONAL PHYSICS 18, 360–375 (1975)

Mathematical and Numerical Analysis of a Differential Equation Model for a Gas Glow Discharge*

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Received January 7, 1975; revised March 19, 1975

A nonlinear characteristic value problem modeling a gas glow discharge is considered. A small parameter solution is found which corresponds to the classical Schottky equilibrium approximation. The operating characteristic is computed using the small parameter solution as a starting point. Uniqueness of the operating characteristic is inferred from a combination of global mathematical analysis and numerical analysis. The operating characteristic is stable. Finally, an observation on constriction is made.

1. PHYSICAL BACKGROUND AND PROBLEM STATEMENT

Electrical discharges in gases have long been studied for their intrinsic interest [6, 11], as well as for applications ranging from fluorescent lighting to high-power lasers. A phenomenon of interest in certain applications is called "constriction." In some cases, as current through the discharge increases past a certain value, the voltage across it drops abruptly, and the glowing portion of the gas constricts, so that it no longer extends across the entire radius of the discharge tube. Hysteresis has been reported in connection with constriction [7], which suggests the possibility of two equilibrium states of the discharge for certain currents. Mathematical models of discharge equilibria, which generally take the form of nonlinear characteristic value problems, are available [5, 10]. It seems an interesting application of the qualitative theory of ordinary differential equations to determine whether or not these problems have two physically meaningful solutions. If they do, the two solutions may explain constriction with the physical phenomena accounted for in the model considered; if they do not, and if the models' unique solutions do not display an abrupt voltage drop, then other physical phenomena must be involved in constriction. The model considered in this paper has unique solutions

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and does not have an abrupt voltage drop. However, it does have a "mild constriction," which is described in the last section. The results presented here are inferred from both mathematical and numerical analysis.

The model considered here can be reduced to the pair of nonlinear differential equations

$$\begin{aligned} (ra(T, E_0)\, T')' + rb(T, u, E_0)\, T &= 0, \\ (rc(T, E_0)\, u')' + rd(T, u, E_0)\, u &= 0, \end{aligned} \qquad \left({}' = \frac{d}{dr}\right), \tag{1.1}$$

subject to the boundary conditions

$$\begin{aligned} T'(0) &= 0, \qquad & u'(0) &= 0, \\ T(1) &= 1, \qquad & u(1) &= 0. \end{aligned} \tag{1.2}$$

In (1.1), r is the dimensionless radius, T is the dimensionless ion temperature (=background gas temperature) and u is a dimensionless form of the sum of the ion and electron pressures ($=n_iT_i + n_eT_e$). The axial field strength E_0 is the characteristic value. The current is proportional to $-T'(1)/E_0$.

This model is the self-consistent equilibrium condition of a nonlinear ionization wave model derived by D. A. Lee, as explained in [10]. It differs from the model of [5, Eqs. (12)–(22)] in that recombination is included and in that a relation of the form

$$T_e = C_1(E_0T)^{4/3} \tag{1.3}$$

replaces [5, Eq. (22)], which has the form

$$T_e = C_2(E_0T). \tag{1.4}$$

In (1.3) and (1.4), C_1 and C_2 are phenomenological constants, while the other quantities have the same meanings as in [5]. Physically, the difference between the two models, aside from recombination, can be shown to arise from different modeling of momentum transfer by collisions between electrons and the background gas.

We will establish the existence of a small parameter solution to (1.1), (1.2) in Section 2, and this solution will be used in Section 3 to compute the operating characteristic. Uniqueness of the operating characteristic will be dealt with in Section 4, and finally Section 5 contains several qualitative results related to the operating characteristic.

The original modeling equations giving rise to (1.1) and (1.2) are listed in the Appendix. The coefficients of (1.1) are

$$\begin{aligned} a(T, E_0) &= 1, \\ b(T, u, E_0) &= PE_0^{10/3}u/(T(1 + DE_0^{4/3}T^{1/3}))^2, \\ c(T, E_0) &= T, \end{aligned}$$

and

$$d(T, u, E_0) = (AE_0^{2/3}T^{7/3}(1 + DE_0^{4/3}T^{1/3})\exp(-B/(E_0T)^{4/3}) - CE_0^{2/3}u)/(T^{4/3}(1 + DE_0^{4/3}T^{1/3}))^2,$$

where A, B, C, D, and P are constants which can be found from the equations in the Appendix and are given there. a, b, c, and d are real analytic functions of T, u, and E_0 for $0 < T < \infty$, $-\infty < u < \infty$, $0 < E_0 < \infty$. We have written the differential equations in the general form (1.1) because (i) one might wish to consider a variable thermal conductivity, $a(T, E_0)$, for the background gas and a momentum transfer collision frequency, essentially $1/c(T, E_0)$, of more general variation; (ii) the small parameter analysis of Section 2 as well as the global analysis of [3] applies to a much broader class of mathematical models than the specific model considered; and (iii) consequently, the numerical computations of Sections 3 and 4 can be expected to succeed for a broader class of differential equation models.

We are interested only in physically meaningful solutions of (1.1) and (1.2). This requires that $u(r) > 0$, $0 \leqslant r < 1$ and $T(r) > 0$, $0 \leqslant r \leqslant 1$. Further, (1.1) is not meaningful if T_e is too large and a generous bound of $T_e < 4$ eV translates as $E_0T < 0.36B^{3/4}$; we let $T_M(E_0) = 0.36B^{3/4}/E_0$. Finally, the validity of (1.1) is based on the plasma's being weakly ionized and in this connection we point out that for the values of E_0 in this paper $u = 150$ corresponds to a value of approximately 10^{-5} for the ratio of the ion number density to the background gas number density.

2. Analyticity and the Small Parameter Solution

Consider the initial value problem for (1.1) with $T(0) = T_0$, $T'(0) = 0$, $u(0) = u_0$, $u'(0) = 0$. A C^2 solution to this problem must satisfy the integral equations

$$\begin{aligned} T(r) &= T_0 - \int_0^r \frac{1}{sa(T(s), E_0)} \int_0^s tb(T(t), u(t), E_0)\, T(t)\, dt\, ds, \\ u(r) &= u_0 - \int_0^r \frac{1}{sc(T(s), E_0)} \int_0^s td(T(t), u(t), E_0)\, u(t)\, dt\, ds, \end{aligned} \tag{2.1}$$

and a C^0 solution to (2.1) is necessarily C^2 and satisfies the initial value problem for (1.1). Using only the Lipschitz continuity of a, b, c, and d and the nonvanishing of a and c in the Picard iteration of (2.1), it can be shown that *solutions to* (2.1) *exist, are unique and continuous* (cf. [4, Chap. 1]). We denote these solutions by $(T(r, T_0, u_0, E_0), u(r, T_0, u_0, E_0))$. We can make use of the analyticity of a, b, c, and d to show further that $(T(r, T_0, u_0, E_0), u(r, T_0, u_0, E_0))$ *is analytic in* (r, T_0, u_0, E_0). The analyticity can be established by the methods used in [4, Section 8, Chap. 1]; the reader is also directed to [2].

The problem to be solved can now be simply stated; namely, find all values of (T_0, u_0, E_0) such that

$$G(T_0, u_0, E_0) = 0,$$

where G is the two-dimensional vector function defined by

$$G_1(T_0, u_0, E_0) = T(1, T_0, u_0, E_0) - 1,$$
$$G_2(T_0, u_0, E_0) = u(1, T_0, u_0, E_0).$$

Once a solution is found, the value of the current is $I(1, T_0, u_0, E_0)$, where

$$I(r, T_0, u_0, E_0) = -QrT'(r, T_0, u_0, E_0)/E_0 \tag{2.2}$$

and Q can be found from the equations in the Appendix and is given there.

Using the property that $b(T, 0, E_0) = 0$ for T, $E_0 > 0$ and uniqueness of solutions, we obtain the trivial solution

$$T(r, T_0, 0, E_0) = T_0, \qquad u(r, T_0, 0, E_0) = 0.$$

In particular, $G(1, 0, E_0) = 0$ for $E_0 > 0$. We shall seek solutions to $G = 0$ by implicit function theorem methods for small values of the parameter u_0.

For the application of the implicit function theorem we have need for the various partial derivatives of the solution $(T(r, T_0, u_0, E_0), u(r, T_0, u_0, E_0))$ when $r = 1$ and $u_0 = 0$. Partial derivatives will be denoted with subscripts. We shall show the computation of $u_{u_0}(r, T_0, 0, E_0)$, the remaining derivatives being found by similar techniques [9]. First, differentiating the second equation of (1.1) with respect to u_0 gives

$$(rcu'_{u_0})' + (rc_T T_{u_0} u')' + r(d_T T_{u_0} + d_u u_{u_0})\, u + r\, du_{u_0} = 0.$$

Setting $u_0 = 0$ in this identity gives the linear equation

$$(rc(T_0, E_0)\, u'_{u_0}(r, T_0, 0, E_0))' + rd(T_0, 0, E_0)\, u_{u_0}(r, T_0, 0, E_0) = 0.$$

Second, $u_{u_0}(0, T_0, 0, E_0) = 1$, $u'_{u_0}(0, T_0, 0, E_0) = 0$. Thus

$$u_{u_0}(r, T_0, 0, E_0) = J_0(\alpha_0 r),$$

where $\alpha_0 = \alpha(T_0, E_0)$ and

$$\alpha(T, E_0) = (d(T, 0, E_0)/c(T, E_0))^{1/2}.$$

The same methods give

$$T_{u_0}(r, T_0, 0, E_0) = \frac{b_u(T_0, 0, E_0)\, T_0}{a(T_0, E_0)} \frac{J_0(\alpha_0 r) - 1}{\alpha_0^2}.$$

Returning to the function $G(T_0, u_0, E_0)$, we can compute the matrix $G'(T_0, u_0, E_0)$ of partial derivatives at $u_0 = 0$, and it is

$$\begin{pmatrix} 1 & T_{u_0}(1, T_0, 0, E_0) & 0 \\ 0 & u_{u_0}(1, T_0, 0, E_0) & 0 \end{pmatrix}.$$

Suppose that at a fixed value of E_0, say $\bar{E}_0$, we have $u_{u_0}(1, 1, 0, \bar{E}_0) \neq 0$. Then the implicit function theorem implies that $G(T_0, u_0, E_0) = 0$ has a unique solution $T_0 = T_0^{\#}(E_0)$, $u_0 = u_0^{\#}(E_0)$ for E_0 in a sufficiently small neighborhood of $\bar{E}_0$, and $T_0^{\#}(\bar{E}_0) = 1$, $u_0^{\#}(\bar{E}_0) = 0$. However, the uniqueness of the solution implies that $T_0^{\#}(E_0) = 1$, $u_0^{\#}(E_0) = 0$ for all E_0 in the aforementioned neighborhood, and these values give the trivial solution to (1.1) and (1.2).

In order to acquire nontrivial solutions for small u_0 we must have $u_{u_0}(1, 1, 0, \bar{E}_0) = 0$ or equivalently, $\alpha(1, \bar{E}_0) = \zeta$, where ζ is a positive zero of J_0. However, ζ must be the smallest positive zero, ζ_0, of J_0, for larger zeros imply that $u(r, T_0, u_0, E_0)$ is negative for some values of r in $[0, 1]$ when (T_0, u_0, E_0) is sufficiently close to $(1, 0, \bar{E}_0)$.

Let us now discuss the equation

$$\alpha(1, E_0) = \zeta_0.$$

This equation has two solutions. However, the larger solutions is such that $E_0 \cdot 1 > 0.36B^{3/4}$. The smaller solution satisfies $E_0 \cdot 1 < 0.36B^{3/4}$, we denote it by E_{0s}, $E_{0s} = 463.6$ and it corresponds to the Schottky equilibrium. Later, we will have use for the function $T = T^{\alpha}(E_0)$ defined as the solution of the equation $\alpha(T, E_0) = \zeta_0$ subject to $E_0 T < 0.36B^{3/4}$. T^{α} is well defined, for one can verify that $\alpha_T > 0$ for $E_0 T < (4B/5)^{3/4}$ and that $\alpha_{E_0} > 0$ for $E_0 T < (2B)^{3/4}$. $T = T^{\alpha}(E_0)$ is strictly decreasing and $T^{\alpha}(E_{0s}) = 1$.

The choice $\bar{E}_0 = E_{0s}$ gives $u_{u_0}(1, 1, 0, \bar{E}_0) = 0$. The function $u_0 J_0(\zeta_0 r)$, when u_0 is small, corresponds to the Schottky equilibrium approximation [12]. The matrix $G'(1, 0, E_{0s})$ is

$$\begin{pmatrix} 1 & -b_u(1, 0, E_{0s})/(a(1, E_{0s})\, \zeta_0^2) & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This matrix is degenerate and the implicit function theorem does not apply.

Fortunately, the degeneracy can be removed by replacing $u(1, T_0, u_0, E_0)$

with $U(1, T_0, u_0, E_0)$, where U is defined by $u(r, T_0, u_0, E_0) = u_0 U(r, T_0, u_0, E_0)$. $(T(r, T_0, u_0, E_0), U(r, T_0, u_0, E_0))$ satisfy the differential equations

$$(ra(T, E_0)\, T')' + rb(T, u_0 U, E_0)\, T = 0,$$
$$(rc(T, E_0)\, U')' + rd(T, u_0 U, E_0)\, U = 0,$$

with the initial conditions

$$T(0) = T_0, \qquad U(0) = 1,$$
$$T'(0) = 0, \qquad U'(0) = 0.$$

And we have the special solution $T(r, T_0, 0, E_0) = T_0$, $U(r, T_0, 0, E_0) = J_0(\alpha_0 r)$.

Let $F(T_0, u_0, E_0)$ be the two-dimensional vector function defined by

$$F_1(T_0, u_0, E_0) = T(1, T_0, u_0, E_0) - 1,$$
$$F_2(T_0, u_0, E_0) = U(1, T_0, u_0, E_0).$$

The equations $F = 0$ and $G = 0$ are equivalent when $u_0 \neq 0$. We have

$$F(1, 0, E_{0s}) = 0.$$

The matrix $F'(1, 0, E_{0s})$ is

$$\begin{pmatrix} 1 & -b_u(1, 0, E_{0s})/(a(1, E_{0s})\, \zeta_0^2) & 0 \\ J_0'(\zeta_0)\, \alpha_T(1, E_{0s}) & U_{u_0}(1, 1, 0, E_{0s}) & J_0'(\zeta_0)\, \alpha_{E_0}(1, E_{0s}) \end{pmatrix}.$$

Now $\alpha_{E_0}(1, E_{0s}) > 0$ and consequently, the submatrix,

$$\begin{pmatrix} F_{1T_0}(1, 0, E_{0s}) & F_{1E_0}(1, 0, E_{0s}) \\ F_{2T_0}(1, 0, E_{0s}) & F_{2E_0}(1, 0, E_{0s}) \end{pmatrix}.$$

of $F'(1, 0, E_{0s})$ is invertible. The implicit function theorem can now be applied and it shows that the equation $F(T_0, u_0, E_0) = 0$ has a solution of the form $T_0 = T_0^*(u_0)$, $E_0 = E_0^*(u_0)$ for u_0 in a sufficiently small neighborhood of $u_0 = 0$, with $T_0^*(0) = 1$, $E_0^*(0) = E_{0s}$. Further, there is a neighborhood of $(1, 0, E_{0s})$ in which the only solutions of $F = 0$ are given by $(T_0^*(u_0), u_0, E_0^*(u_0))$ for u_0 in the aforementioned neighborhood. T_0^* and E_0^* are analytic. The current is also an analytic function of u_0 and it is given by $I = I^*(u_0)$, where

$$I^*(u_0) = -QT'(1, T_0^*(u_0), u_0, E_0^*(u_0))/E_0^*(u_0)$$

and $I^*(0) = 0$.

Now $U_{u_0}(r, T_0, 0, E_0)$ can be found by the techniques above together with the method of variation of constants, and for $0 < \alpha_0 r < \zeta_0$ it is given by

$$U_{u_0}(r, T_0, 0, E_0)$$
$$= -J_0(\alpha_0 r) \int_0^r \frac{1}{tJ_0^2(\alpha_0 t)} \int_0^t sJ_0(\alpha_0 s) \left\{ \frac{b_u(T_0, 0, E_0)\, T_0}{a(T_0, E_0)} \left[\frac{c_T(T_0, E_0)}{c(T_0, E_0)} J_0'^2(\alpha_0 s) \right.\right.$$
$$\left.\left. + \frac{2\alpha_T(T_0, E_0)}{\alpha_0} J_0(\alpha_0 s)(J_0(\alpha_0 s) - 1) \right] + \frac{d_u(T_0, 0, E_0)}{c(T_0, E_0)} J_0^2(\alpha_0 s) \right\} ds\, dt,$$

with

$$U_{u_0}(1, 1, 0, E_{0s})$$
$$= -\frac{b_u(1, 0, E_{0s})}{a(1, E_{0s})} \alpha_T(1, E_{0s}) \frac{J_0'(\zeta_0)}{\zeta_0^2} + \left[\frac{d_u(1, 0, E_{0s})}{c(1, E_{0s})} + \frac{b_u(1, 0, E_{0s})}{a(1, E_{0s})} \right.$$
$$\left. \times \left(\frac{1}{2} \frac{c_T(1, E_{0s})}{c(1, E_{0s})} + 2 \frac{\alpha_T(1, E_{0s})}{\zeta_0} \right) \right] \left(\int_0^{\zeta_0} tJ_0^3(t)\, dt \right) (\zeta_0^3 J_0'(\zeta_0))^{-1}.$$

With the last formula we can compute the derivatives of T_0^*, E_0^*, and I^* w.r.t. u_0 at $u_0 = 0$ and they are

$$T_0^{*\prime}(0) = b_u(1, 0, E_{0s})/\zeta_0^2 a(1, E_{0s}),$$
$$E_0^{*\prime}(0) = -\left[\frac{d_u(1, 0, E_{0s})}{c(1, E_{0s})} + \frac{b_u(1, 0, E_{0s})}{a(1, E_{0s})} \left(\frac{1}{2} \frac{c_T(1, E_{0s})}{c(1, E_{0s})} + \frac{2\alpha_T(1, E_{0s})}{\zeta_0} \right) \right]$$
$$\times \left(\int_0^{\zeta_0} tJ_0^3(t)\, dt \right) (\zeta_0^3 J_0'^2(\zeta_0))^{-1} (\alpha_{E_0}(1, E_{0s}))^{-1},$$
$$I^{*\prime}(0) = -Q[b_u(1, 0, E_{0s})/a(1, E_{0s})\, E_{0s}][J_0'(\zeta_0)/\zeta_0].$$

We note that $I^{*\prime}(0) > 0$. For the modeling constants chosen $E_0^{*\prime}(0)$ $(=17.2)$ is also positive.

Since $I^{*\prime}(0) > 0$, the set of points $(I^*(u_0), E_0^*(u_0))$ for small $u_0 \geqslant 0$ in the $I - E_0$ plane can be expressed as the graph of a function of I. This graph has the property that $dE_0/dI\,|_{I=0} = E_0^{*\prime}(0)/I^{*\prime}(0) > 0$, i.e., the operating characteristic has positive slope for small currents. This property is in agreement with certain experiments performed at the Aerospace Research Laboratories [1]. Observing that $b_u > 0$, we find that the expression for $E_0^{*\prime}(0)$ indicates that the positive slope property requires a nonzero recombination rate, essentially d_u, a feature not included in the model of Ecker and Zöller. *Now the small parameter analysis can also be applied successfully to* (1.1) *for the case of a radiation–convection–conduction boundary condition to show the existence of a small parameter solution.* Using a wall emissivity of 0.9 along with a convective heat transfer coefficient proportional to $(T(1) - 1)^{1/4}$, $dE_0/dI\,|_{I=0}$ was computed and found to be negative, which also agrees with experiment.

It is of some value, as will be seen in Section 4, to consider the equations

$$T(1, T_0, u_0, E_0) - 1 = 0, \qquad U(1, T_0, u_0, E_0) = 0$$

separately. From $T(1, 1, 0, E_0) - 1 = 0$, $T_{T_0}(1, 1, 0, E_0) = 1$, and $T_{u_0}(1, 1, 0, E_0) < 0$, we see that a solution $u_0 = u_0^{\#}(T_0, E_0)$ exists to the first equation for $|T_0 - 1|$ sufficiently small; $u_0^{\#}(1, E_0) = 0$ and $u_0^{\#}$ has positive slope at $T_0 = 1$. Thus for a given E_0, there is a curve of positive slope in the $T - u$ plane emanating from $T = 1$, $u = 0$ with the property that each point of the curve is an initial value such that $T(1, T_0, u_0, E_0) = 1$. Concerning the second equation, let $\bar{T}_0 = T^{\alpha}(E_0)$. Then $U(1, \bar{T}_0, 0, E_0) = 0$ and $U_{T_0}(1, \bar{T}_0, 0, E_0) = J_0'(\zeta_0)\,\alpha_T(\bar{T}_0, E_0) < 0$. A separate calculation shows that $U_{u_0}(1, \bar{T}_0, 0, E_0) > 0$ at least for $300 \leqslant E_0 \leqslant 500$ for the modeling constants chosen. This calculation is very tedious and uses a formula for $U_{u_0}(1, \bar{T}_0, 0, E_0)$ that is similar to the formula for $U_{u_0}(1, 1, 0, E_{0s})$. For this range of E_0 we see that a solution $u_0 = u_0^{\#\#}(T_0, E_0)$ exists to the second equation for $|T_0 - \bar{T}_0|$ sufficiently small; $u_0^{\#\#}(\bar{T}_0, E_0) = 0$ and $u_0^{\#\#}$ has positive slope at $T_0 = \bar{T}_0$. For such an E_0, we have that there is a curve of positive slope in the $T - u$ plane emanating from $T = \bar{T}_0$, $u = 0$ with the property that each point of the curve is an initial value such that $U(1, T_0, u_0, E_0) = 0$. We note further that $\bar{T}_0 > 1$ when $E_0 < E_{0s}$ and $\bar{T}_0 < 1$ when $E_0 > E_{0s}$.

3. Computation of the Operating Characteristic

The equation $F(T_0, u_0, E_0) = 0$ was solved for $T_0 = T_0^*(u_0)$, $E_0 = E_0^*(u_0)$ by fixing u_0 and using Newton iteration. The necessary partial derivatives were found by integrating the system of ordinary differential equations satisfied by the set of partial derivatives through use of DIFSUB by Gear [8]. The initial guesses for the Newton iteration were provided as follows. For $u_0 = 0$ and 0.1, $T_0 = 1$ and $E_0 = 463.5$ were the initial guesses. Once $T_0^*(0)$, $E_0^*(0)$, $T_0^*(0.1)$, and $E_0^*(0.1)$ were found, linear extrapolation on T_0^* and E_0^* was perfomed to provide initial guesses for the range $0 \leqslant u_0 \leqslant 10$. At $u_0 = 10$ we are beyond the maximum of the operating characteristic. For $u_0 > 10$, the initial guesses for $T_0^*(u_0)$ and $E_0^*(u_0)$ were quite effectively extrapolated from a fourth-order Lagrange interpolating polynomial.

The computation was carried out for $0 \leqslant u_0 \leqslant 467$ with no indication that the Newton iteration would fail for larger values of u_0. The qualitative behavior of the solutions $T_0 = T_0^*(u_0)$ and $E_0 = E_0^*(u_0)$ as well as $I = I^*(u_0)$ is indicated in Figs. 1(a)–(c). We denote the maximum value of E_0 in Figs. 1(b) and 2 by E_{0M},

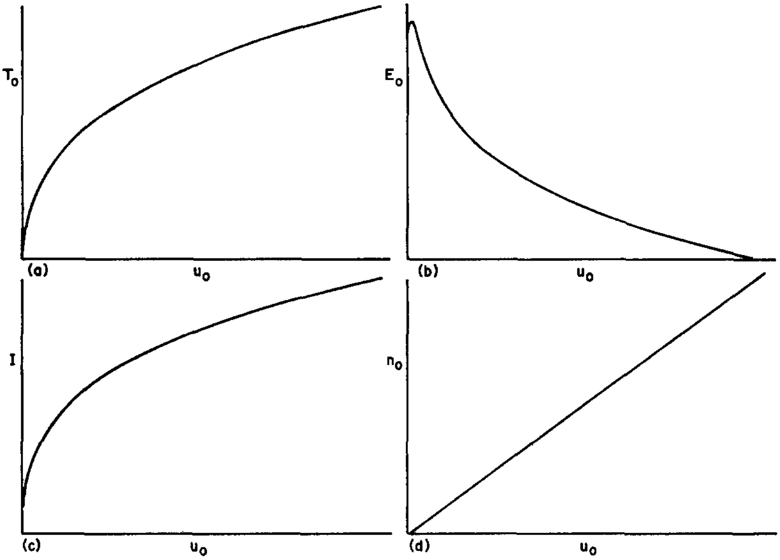

FIG. 1. The qualitative behavior of the solutions $T_0 = T_0^*(u_0)$, $E_0 = E_0^*(u_0)$ of $F(T_0, u_0, E_0) = 0$, as well as the corresponding current $I = I^*(u_0)$ and the center number density $n_0 = n^*(u_0)$.

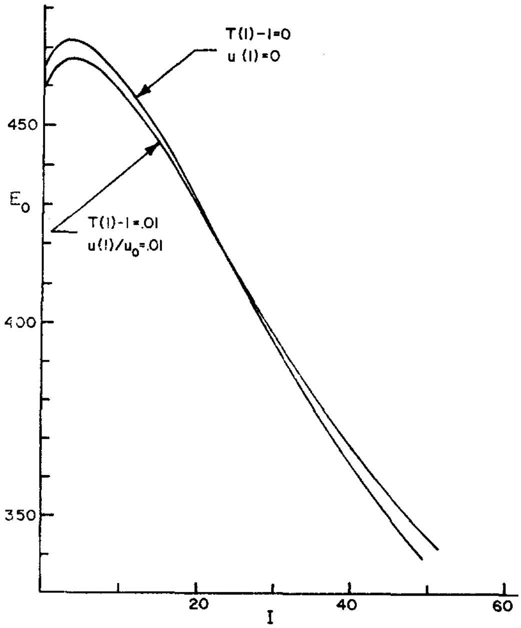

FIG. 2. The constant wall temperature characteristic for (1.1). E_0 is in volts/meter and I is in milliamps.

$E_{0M} = 471.8$. Computations were not continued beyond $u_0 = 467$ because the validity of the equations becomes tenuous and because significant changes in the solutions to $F = 0$ were not expected.

We mention that if the initial guesses for the Newton iteration were not close to the solutions of $F = 0$, then the differential equations for the partial derivatives were difficult to integrate with nonstiff methods; stiff methods were not used.

The Jacobian matrix of the map $(T_0, u_0, E_0) \to (T(1, T_0, u_0, E_0), u(1, T_0, u_0, E_0), I(1, T_0, u_0, E_0))$ was computed along the solution $T_0 = T_0^*(u_0)$, $E_0 = E_0^*(u_0)$ and it was found to be invertible for $u_0 > 0$. An important implication is that the operating characteristic is locally unique. We will deal with global uniqueness in the next section.

4. Number of Solutions

A numerical search procedure indicates that the curves of Figs. 1(a) and (b) provide the only solutions of $F = 0$. The procedure utilizes two results of [3]. The first result is that two functions of T_0 and E_0 are found, $u = \mathscr{U}(T_0, E_0)$, $u = \mathscr{L}(T_0, E_0)$, such that if for a given E_0, (T_0, u_0) is an initial value corresponding to a nontrivial solution to (1.1) and (1.2), then

$$1 < T_0 < T_M(E_0)$$

and

$$\mathscr{L}(T_0, E_0) < u_0 < \mathscr{U}(T_0, E_0).$$

Thus, one need only consider initial values satisfying these inequalities.

We will consider initial values lying in a somewhat large set. To describe that set we need the functions $u = h(T, E_0)$ and $u = k(T, E_0)$ defined respectively by the equations

$$d(T, u, E_0) = 0,$$

and

$$f_T(T, u, E_0)\, g(T, E_0) + f_u(T, u, E_0)\, f(T, u, E_0) = 0,$$

where $f = d(T, u, E_0)\, u/(b(T, u, E_0)\, T)$ and $g = c(T, E_0)/a(T, E_0)$. These equations are linear in u and give explicit expressions for h and k. We easily verify that $0 < h(T, E_0), k(T, E_0) < h(T, E_0)$, and $d(T, u, E_0)(u - h(T, E_0)) < 0$ if $u \neq h(T, E_0)$.

It is the case that $\mathscr{U}(T_0, E_0) < h(T_0, E_0)$. The larger set is

$$\begin{gathered} 1 < T_0 < T_M(E_0), \\ \min(k(T_0, E_0), h(T_0, E_0)(1 - 1/(1 + C \ln T_0/(PE_0^{8/3} T_0^{8/3}(1 + DE_0^{4/3} T_0^{1/3}))))) \\ < u_0 < h(T_0, E_0), \end{gathered} \tag{4.1}$$

where the second function on the left-hand side is $\mathscr{L}(T_0, E_0)$ when $\mathscr{L}(T_0, E_0) \leqslant k(T_0, E_0)$. For a given E_0, we denote the set of points (T_0, u_0) satisfying (4.1) as well as $0 < u_0 < u_{0,\max}$ by $R(E_0, u_{0,\max})$.

The second result of [3] is that when (T_0, u_0) is an initial value for a solution to (1.1) and (1.2) for a given E_0, then $d(T_0, u_0, E_0) > 0$ and $u'(r, T_0, u_0, E_0) < 0$, $0 < r \leqslant 1$. An integration of the first equation in (1.1) shows that $T'(r, T_0, u_0, E_0) < 0, 0 < r \leqslant 1$ also. Consequently, $y_1 = T, y_2 = T'$, $y_3 = u$, $y_4 = u'$ is a solution to

$$\begin{aligned}
y_1' &= y_2^*, \\
y_2' &= -\left(\frac{a_T(y_1^*, E_0)}{a(y_1^*, E_0)} y_2^* + \frac{1}{r}\right) y_2^* - \frac{b(y_1^*, y_3^*, E_0)\, y_1^*}{a(y_1^*, E_0)}, \\
y_3' &= y_4^*, \\
y_4' &= -\left(\frac{c_T(y_1^*, E_0)}{c(y_1^*, E_0)} y_2^* + \frac{1}{r}\right) y_4^* - \frac{d(y_1^*, y_3^*, E_0)\, y_1^*}{c(y_1^*, E_0)},
\end{aligned} \tag{4.2}$$

for $r > 0$, where $y_1^* = \max(1, y_1)$, $y_2^* = \min(0, y_2)$, $y_3^* = \max(0, y_3)$, $y_4^* = \min(0, y_4)$ and

$$\begin{aligned}
y_1'(0) &= 0, \\
y_2'(0) &= -\tfrac{1}{2}[b(T_0, u_0, E_0)\, T_0/a(T_0, E_0)], \\
y_3'(0) &= 0, \\
y_4'(0) &= -\tfrac{1}{2}[d(T_0, u_0, E_0)\, u_0/c(T_0, E_0)],
\end{aligned} \tag{4.3}$$

satisfying the boundary conditions

$$\begin{aligned}
y_2(0) &= 0, \qquad y_1(1) = 1, \\
y_4(0) &= 0, \qquad y_3(1) = 0.
\end{aligned}$$

Consider the initial value problem for (4.2) and (4.3) with $y_1(0) = T_0$, $y_2(0) = 0$, $y_3(0) = u_0$, $y_4(0) = 0$, where the initial values are restricted to $1 < T_0$, $0 < u_0$, and $d(T_0, u_0, E_0) > 0$. This problem has a well-defined C^1 solution. For it can be shown that $y_1(r, T_0, u_0, E_0) = T(r, T_0, u_0, E_0)$, $y_2(r, T_0, u_0, E_0) = T'(r, T_0, u_0, E_0)$, $y_3(r, T_0, u_0, E_0) = u(r, T_0, u_0, E_0)$, $y_4(r, T_0, u_0, E_0) = u'(r, T_0, u_0, E_0)$ for small $r \geqslant 0$, while for any set on which r is bounded away from zero the right-hand side of (4.2) is Lipschitz continuous. Further, it is elementary to show that these solutions exist for all $r \geqslant 0$. In contrast, numerical analysis indicates that the solutions of (1.1) do not exist for all $r \geqslant 0$. Any initial values leading to a solution to (1.1) and (1.2) for a given E_0 must also give a solution to

$$y_1(1, T_0, u_0, E_0) - 1 = 0, \tag{4.4}$$

$$y_3(1, T_0, u_0, E_0) = 0, \tag{4.5}$$

and (T_0, u_0, E_0) must satisfy (4.1). A particular value of considering the equivalent problem (4.2)–(4.5) is that the solutions to (4.2) and (4.3) are much easier to compute than those for (1.1).

The numerical search procedure consisted of first choosing an adequately large value of $u_{0,\max}$ for a given E_0, e.g., from Fig. 1(b), and then superimposing a uniform rectangular grid on $R(E_0, u_{0,\max})$ and second computing $y_1(1, T_0, u_0, E_0)$, $y_3(1, T_0, u_0, E_0)$ at each grid point. The numerical results indicated that the set of points (T_0, u_0) for a given E_0 such that $y_1(1, T_0, u_0, E_0) - 1 = 0$ is the graph of a strictly increasing function of T_0 emanating from $T_0 = 1$, $u_0 = 0$ and that $y_1(1, T_0, u_0, E_0)$ is strictly increasing in T_0 and strictly decreasing in u_0. The results also indicated that the set of points (T_0, u_0) such that $y_3(1, T_0, u_0, E_0) = 0$ is the graph of a strictly increasing function of T_0 emanating from a point $T_0 = \overline{T}_0$,

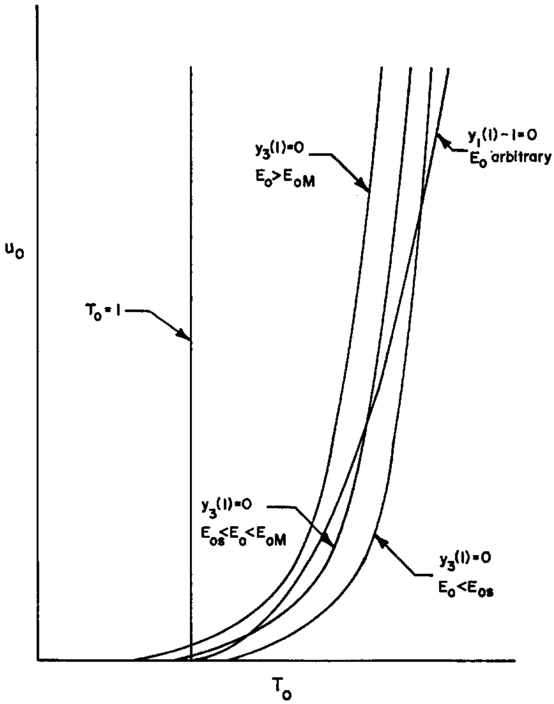

FIG. 3. The nature of the intersections of the curves $y_1(1, T_0, u_0, E_0) - 1 = 0$ and $y_3(1, T_0, u_0, E_0) = 0$.

$u_0 = 0$ where $\bar{T}_0 < 1$ when $E_0 > E_{0s}$ and $\bar{T}_0 > 1$ when $E_0 < E_{0s}$, and that $y_3(1, T_0, u_0, E_0)$ is strictly decreasing in T_0 and strictly increasing in u_0 (when $E_0 < E_{0s}$ it can be shown that $y_1 = T$, $y_2 = T'$, $y_3 = u$, $y_4 = u'$ for u_0 small, and consequently, $\bar{T}_0 = T^\alpha(E_0)$). The computation further indicated that the solution curves for (4.4) and for (4.5) do not cross if $E_0 > E_{0M}$, that they cross twice if $E_{0s} < E_0 < E_{0M}$, and that they cross once if $E_0 < E_{0s}$ (see Fig. 3). When the grid output indicated a set of points (T_0, u_0) where a simultaneous solution to (4.4) and (4.5) might lie, Eqs. (4.4) and (4.5) were separately iterated to further isolate zeros. The numerical results obtained further reinforced the conclusions represented in Fig. 3.

Lastly, we will make a short note on solutions in a neighborhood of $u_0 = 0$. In [3] it is established that

$$u_0 J_0 \left(\left(\frac{d(T(r_0, T_0, u_0, E_0), 0, E_0)}{c(T(r_0, T_0, u_0, E_0), E_0)} \right)^{1/2} r \right) < u(r, T_0, u_0, E_0)$$

for $0 \leqslant r \leqslant r_0$, where r_0 is the first zero of $u(r, T_0, u_0, E_0)$ provided that $0 < u_0 \leqslant k(T_0, E_0)$. Computation shows that $k(1, E_{0s}) > 0$. If (T_0, u_0) gives a solution to (1.1) and (1.2) for a given E_0, then $r_0 = 1$ and

$$u_0 J_0((d(1, 0, E_0)/c(1, E_0))^{1/2}) < 0.$$

In particular, $\alpha(1, E_0) > \zeta_0$, or $E_0 > E_{0s}$. Thus, there are no solutions to (1.1) and (1.2) for $0 < u_0 \leqslant k(T_0, E_0)$ and $E_0 \leqslant E_{0s}$, and, with regard to the operating characteristic, this fact implies that $E_0 > E_{0s}$ for small positive u_0 and, in turn, for small positive currents.

5. Qualitative Aspects

For the values of T_0, u_0, E_0 considered in this paper, Theorem 3.1 of [3] applies and it states that the number density corresponding to a solution of (1.1) and (1.2) is a strictly decreasing function of r. Another observation is that the ambipolar current is positive for $0 < r \leqslant 1$, for it is proportional to $-c(T, E_0)\, u'$.

Next, we asked whether or not small changes in the wall conditions, i.e., in $T(1, T_0, u_0, E_0)$, $u(1, T_0, u_0, E_0)$, and $I(1, T_0, u_0, E_0)$ result in large shifts off of the operating characteristic, especially at larger currents. The answer is no for this model. The eigenvector corresponding to the largest eigenvalue of the Jacobian matrix of the inverse map $(T(1, T_0, u_0, E_0), u(1, T_0, u_0, E_0), I(1, T_0, u_0, E_0)) \to (T_0, u_0, E_0)$ was computed along the solution path $(T_0{}^*(u_0), u_0, E_0{}^*(u_0))$, $0 \leqslant u_0 \leqslant 467$, and it was found that this vector was nearly parallel to the tangent vector of the solution path. Consequently, if a change in $(T(1, T_0, u_0, E_0)$, $u(1, T_0, u_0, E_0), I(1, T_0, u_0, E_0))$ was made in the direction which would give rise to the largest change in (T_0, u_0, E_0), then this change occurs in a direction nearly parallel to the path $(T_0^*(u_0), u_0, E_0^*(u_0))$, $0 \leqslant u_0 \leqslant 467$, and thus gives an operating point close to the operating characteristic. The second eigenvalue of the aforementioned matrix was of moderate magnitude while the third became quite small as u_0 increased.

The effect of small perturbations in $T(1, T_0, u_0, E_0)$ and $U(1, T_0, u_0, E_0) = u(1, T_0, u_0, E_0)/u_0$ along constant current lines was also investigated. Let e be a two-dimensional unit vector and δ a small scalar parameter. Consider the equation

$$H(T_0, u_0, E_0, \delta, e) = 0,$$

where $H(T_0, u_0, E_0, \delta, e) = F(T_0, u_0, E_0) - \delta e$. By comparing H to F one can see that $H = 0$ has a solution $T_0 = T_0^{\#}(u_0, \delta, e)$, $E_0 = E_0^{\#}(u_0, \delta, e)$, for all sufficiently small δ, such that $T_0^{\#}(u_0, 0, e) = T_0^*(u_0)$, $E_0^{\#}(u_0, 0, e) = E_0^*(u_0)$. Corresponding to $T_0^{\#}$ and $E_0^{\#}$ we obtain the current $I = I^{\#}(u_0, \delta, e)$ from (2.2). The derivatives $E_{0\delta}^{\#}(u_0, 0, e)$, $I_\delta^{\#}(u_0, 0, e)$ are directly computable. It was found that if e was chosen so as to make $I_\delta^{\#}(u_0, 0, e) = 0$, then $|E_{0\delta}^{\#}(u_0, 0, e)/E_0^*(u_0)| < 1$. For the purpose of reinforcing this result, (1.1) was solved subject to the wall conditions $T(1) = 1.01$, $u(1)/u_0 = 0.01$ and the resulting operating characteristic is plotted in Fig. 2.

Finally, we will make an observation related to the constriction of the visible part of the position column. The number density of ions (and electrons) is given by $n = u/(T + D(E_0T)^{4/3})$. Consequently, the number density at the center of the column is a function of u_0, $n_0 = n^*(u_0)$. Its behavior is very nearly linear as indicated in Fig. 1(d). In contrast, Figs. 1(b) and (c) indicate that the curves $E_0 = E_0^*(u_0)$ and $I = I^*(u_0)$ become flatter as u_0 grows. Thus, the maximum n_0 of the number density profile increases linearly with increasing u_0 while motion along the operating characteristic becomes slower. Reference [10] contains further discussion on the profiles of the number density.

Appendix

1. *Original Equations*

$$(rJ)' = r[Z(T_e)\, n_e - \rho(T_e)\, n_e n_i]$$
$$(n_e T_e)' = -q_0 n_e E_r - m_e \nu_e J$$
$$(n_i T_i)' = q_0 n_i E_r - m_i \nu_i J$$
$$(r T_n')' = -r \nu_{e0} n_e T_e / K_0$$
$$E_0 I = -2\pi(10^3)\, K_0 R T_n'(R)$$
$$\nu_{e0} T_e = m_e \nu_e (q_0 E_0 / m_e \nu_e)^2$$
$$\nu_{e0} = \tilde{\nu}_{e0} T_e^{1/2}/T_n, \qquad \nu_e = \tilde{\nu}_e / T_n, \qquad \nu_i = \tilde{\nu}_i / T_n$$

$Z(T_e) = c_1 T_e^{1/2} \exp(-c_2/T_e)$	$\rho(T_e) = \beta T_e^{-1/2}$
$n_i = n_e = n$	$T_n = T_i = T$
$n'(0) = 0$	$T'(0) = 0$
$n(R) = 0$	$T(R) = T_w$
subscript i = ion	subscript n = neutral particle
subscript e = electron	J = ambipolar current
T = temperature	n = number density
q_0 = electron charge	I = current
K_0 = thermal conductivity	E_0 = axial electric field
R = radius of discharge	T_w = surface temperature
m = mass	Z = ionization frequency

ρ = electron recombination frequency

E_r = radial electric field

ν_e, ν_i = electron, ion momentum transfer collision frequencies

ν_{e0} = energy transfer collision frequency

c_1, c_2, β, $\tilde{\nu}_{e0}$, $\tilde{\nu}_e$, $\tilde{\nu}_i$ are constants

2. *Reference Values*

$$r_{\text{ref}} = R, \qquad T_{\text{ref}} = T_w, \qquad n_{\text{ref}} = 10^{17},$$
$$u_{\text{ref}} = n_{\text{ref}}(q_0{}^2 E_0{}^2 T_w{}^2 / m_e \tilde{\nu}_e \tilde{\nu}_{e0})^{2/3}$$

3. *The Constants A, B, C, D, P, and Q*

$$A = \frac{(m_e\tilde{\nu}_e + m_i\tilde{\nu}_i)\, c_1 R^2}{T_{\text{ref}}^{4/3}} \left(\frac{q_0{}^2}{m_e\tilde{\nu}_e\tilde{\nu}_{e0}}\right)^{1/3}$$

$$B = \frac{c_2}{T_{\text{ref}}^{4/3}} \left(\frac{q_0{}^2}{m_e\tilde{\nu}_e\tilde{\nu}_{e0}}\right)^{-2/3}$$

$$C = \frac{(m_e\tilde{\nu}_e + m_i\tilde{\nu}_i)\, \beta R^2 n_{\text{ref}}}{T_{\text{ref}}^{7/3}} \left(\frac{q_0{}^2}{m_e\tilde{\nu}_e\tilde{\nu}_{e0}}\right)^{1/3}$$

$$D = T_{\text{ref}}^{1/3} \left(\frac{q_0{}^2}{m_e\tilde{\nu}_e\tilde{\nu}_{e0}}\right)^{2/3}$$

$$P = \frac{R^2 n_{\text{ref}} T_{\text{ref}}^{1/3} \tilde{\nu}_{e0}}{K_0} \left(\frac{q_0{}^2}{m_e\tilde{\nu}_e\tilde{\nu}_{e0}}\right)^{5/3}$$

$$Q = -2\pi(10^3)\, K_0 T_{\text{ref}}$$

ACKNOWLEDGMENT

The author expresses great appreciation to his NRC Scientific Advisor, Dr. David A. Lee, whose interest in discharge plasmas inspired the author's research.

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